Absorbing phase transition with a conserved field

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(Received 9 March 2005; published 28 June 2005)

We study a lattice gas model where the number of particles is conserved during dynamical process. Our model shows a continuous phase transition from a fluctuating phase to two symmetric absorbing states at the critical point in one dimension. We conjecture the values of the critical exponents characterizing the phase transition of our model. We show that the obtained values are in good agreement with those estimated from computer simulations. The critical exponents indicate that our model exhibits an absorbing phase transition which is different from the known ones.

DOI: 10.1103/PhysRevE.71.066129 PACS number(s): 64.60.Ak, 82.40.−g, 05.70.Ln, 64.60.Ht

Nonequilibrium phase transitions from a fluctuating phase to one or several absorbing states occur in various reaction-diffusion processes [1–3]. The critical behavior of nonequilibrium phase transitions is similar to that of equilibrium phase transitions in many respects. Therefore, the critical behavior of nonequilibrium phase transitions can be understood well by using the concept of scale invariance introduced to understand equilibrium phase transitions.

From studies of critical exponents characterizing phase transitions, it is believed that nonequilibrium phase transitions occurring in various reaction-diffusion processes can be categorized into several universality classes. Among these classes, the directed percolation (DP) class is the most well-known one [2,3]. Nonequilibrium phase transitions from a fluctuating phase to a *single* absorbing state generally belong to the DP class. Most of absorbing transitions with many absorbing states are also known to fall into the DP class [4]. However, if there exist long-range interactions or additional symmetries such as symmetric absorbing states, parity conservation, etc., absorbing phase transitions may not fall into the DP class. In those cases, a well known class is the parity conserving (PC) class [5–8]. The PC class appears in reaction-diffusion processes where the number of particles is conserved modulo 2 during the dynamical process. The PC class also occurs in absorbing phase transitions with two symmetric absorbing states [9,11]. The pair contact process with diffusion (PCPD) class appears in the reaction-diffusion process where two particles have to meet at two adjacent places in order to create a new particle or particles [10].

In reaction-diffusion processes exhibiting an absorbing phase transition, the stationary particle density ρ_s depends on a particle creation and annihilation rate. If the particle creation rate p is greater than a certain critical value p_c, ρ_s has a nonzero constant value, but if $p<p_c$, ρ_s is zero. The order parameter ρ_s vanishes algebraically as $\rho_s \sim (p-p_c)^{\beta}$ close to a continuous transition point. The ultimate survival probability P_∞ which a randomly selected site belongs to an infinite cluster scales as $P_\infty \sim (p-p_c)^{\beta'}$. β and β' coincide for DP but they do not in general [3,12]. Nonequilibrium phase transitions are characterized by a spatial and a temporal correlation length. The spatial and temporal correlation lengths diverge as $\xi_\perp \sim |p-p_c|^{-\nu_\perp}$ and $\xi_\parallel \sim |p-p_c|^{-\nu_\parallel}$ close to the transition point. These two correlation length scales are related by $\xi_\parallel \sim \xi_\perp^z$, where $z=\nu_\parallel/\nu_\perp$ is called the dynamic exponent. $\beta, \beta', \nu_\parallel, \nu_\perp$ are a basic set of critical exponents characterizing the universality class of a given reaction-diffusion process [3].

Recently, Rossi *et al.* introduced a conserved lattice gas (CLG) model [13], where the number of particles is conserved during the dynamical process. In the model, there is no particle creation and annihilation process. At each time step, a randomly selected particle hops to a vacant nearest neighbor site provided that one of its nearest neighbor sites is already occupied. If an active particle cannot find a vacant nearest neighbor site, the particle is immobile. A particle is called active if at least one of its neighboring sites is occupied by a particle. There is no creation process of an active particle in the CLG model because there is no creation process of a new particle in the model. Hence the number of active particles continuously decreases and eventually reaches a steady state as time elapses. The stationary active particle density ρ_s in the CLG model depends on the particle density n in the initial state. If the particle density n is greater than a critical density n_c, ρ_s has a constant nonzero value. But if $n<n_c$, ρ_s vanishes. The CLG model shows an absorbing phase transition into many absorbing states at n_c. One can apply the same scaling concept, which is used to understand nonequilibrium absorbing phase transitions without a conserved field, to the CLG model. The critical exponents obtained from the simulation of the CLG model have different values from those expected from the known universality classes such as DP, PC, and PCPD class [13–15]. Rossi *et al.* conjectured that the stochastic models such as conserved threshold transfer processes and stochastic sandpile models [13,16,17], where the order parameter is coupled to a nondiffusive conserved field, define a unique universality class [13,18].

From the fact that the universality class of reaction-diffusion processes without a conserved field is determined according to symmetry properties of the absorbing states,

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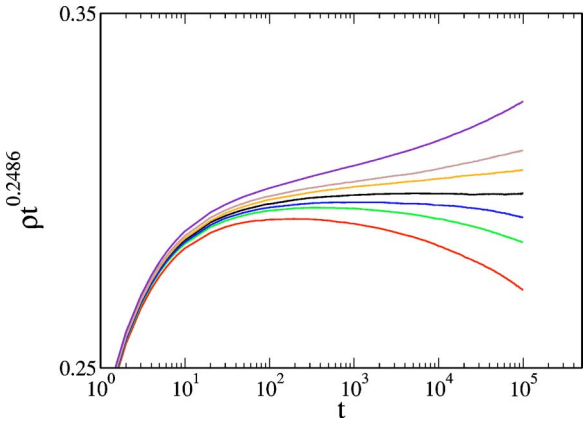

FIG. 1. The density of particles, $\rho(t)$, as a function of time t for $p=0.5008$, 0.5004, 0.5002, 0.5000, 0.4998, 0.4996, and 0.4992 from top to bottom, averaged over 100 runs on a system with size $L=10^6$.

naively one can infer that, if a lattice gas model with a conservative field exhibits a phase transition to two symmetric absorbing states, the model might belong to another universality class which is different from the original CLG model introduced by Rossi *et al.* Therefore, it would be interesting to study a CLG model exhibiting a phase transition to two symmetric absorbing states.

In this paper, we introduce a simple conserved lattice gas model which shows a nonequilibrium phase transition to two exactly symmetric absorbing states at criticality. During the dynamical process, the total number of particles in our model is conserved in the same way as in the original CLG model. As expected, our model shows different scaling behavior from that of the original CLG model having many absorbing states. We can conjecture the values of the critical exponents characterizing the dynamical behavior of our model in one dimension. The conjectured values of the critical exponents indicate that our model belongs to a different universality class unknown until now.

We consider a simple conserved lattice gas model defined on a one-dimensional lattice with linear size L and periodic boundary conditions. Initially the lattice is completely filled with two species of particles (A and B). The ratio $r_{AB}=N_A/N_B$, where N_A and N_B denote the total number of A and B particles, is fixed at a value between 0 to 1 during computer simulation. Initially, A and B particles are randomly distributed on the system with a fixed value of r_{AB}. We define a particle at a given site as an active particle if at least one of its neighboring sites is occupied by a particle of the same species. If all of its neighboring sites are occupied by particles of the different species, the particle is called inactive. Each time we choose a site randomly. If the particle at the selected site is an active one and one of its neighboring sites is occupied by a particle of the different species, the two particles are exchanged instantaneously. The process corresponds to the following schemes

$$AAB \rightarrow ABA$$
$$BAA \rightarrow ABA$$
$$BBA \rightarrow BAB$$
$$ABB \rightarrow BAB,$$

where the particle in the middle in each process is the selected one. In our model, there is neither creation nor annihilation of particles. Just a randomly selected particle can change its position by exchanging two particles. If the number of A and B particles is initially the same, our model reaches two symmetric absorbing states in the long time limit such as

$$ABAB \cdots ABAB$$
$$BABA \cdots BABA.$$

However, if $N_A \neq N_B$, there always exist active particles in our model.

Our model can be easily mapped onto a well known process (AB process) [19,20]: $AB \rightarrow \emptyset$. Let us define ●, ○, and $\emptyset$ as follows: ● and ○ denote the middle of AA and BB particles, respectively. That is, $A \bullet A$ and $B \circ B$. The symbol $\emptyset$ denotes the middle of AB and BA particles, i.e., $A\emptyset B$ and $B\emptyset A$. Let us consider the following processes, where the second particle on the left side is a randomly selected one in each process,

$$AABA \rightarrow ABAA$$
$$AABB \rightarrow ABAB$$
$$BBAA \rightarrow BABA$$
$$BBAB \rightarrow BABB.$$

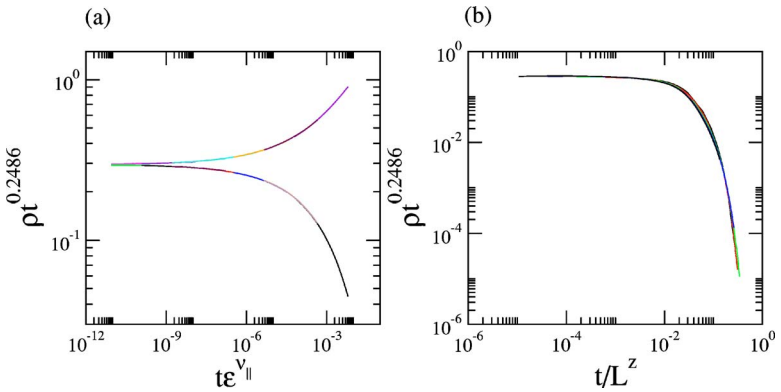

FIG. 2. (a) Data collapse for off-critical simulations according to the scaling form (2) for $L=10^6$ and $\nu_\parallel=3.8$ averaged over 1000 runs. Upper data are for $p=0.5001$, 0.5002,…, 0.5128 and lower data are for $p=0.4999$, 0.4998,…, 0.4872. (b) Finite size data collapse according to (3) for $z=1.95$ and system sizes $L=64$, 128, 256, 512, and 1024 averaged over 50000 runs.

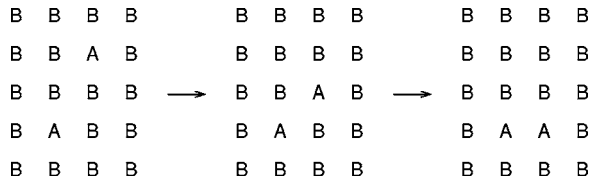

FIG. 3. Two successive exchange processes of two particles, A and B, in two dimensions. Two A particles surrounded by B particles meet each other after the exchange processes. In one dimension, A particles surrounded by B particles cannot meet each other at all and vice versa.

In the above processes, we considered just the cases that a selected particle can be exchanged with a particle on its right side. We can rewrite the above processes by using the symbols ●, ○, and ∅:

$$\bullet\emptyset\emptyset \rightarrow \emptyset\emptyset\bullet$$
$$\bullet\emptyset\circ \rightarrow \emptyset\emptyset\emptyset$$
$$\circ\emptyset\bullet \rightarrow \emptyset\emptyset\emptyset$$
$$\circ\emptyset\emptyset \rightarrow \emptyset\emptyset\circ .$$

Using this notation, the processes can be interpreted as follows: ● or ○ on the most left side is a randomly selected particle in each process. ∅ denotes an empty site. The selected particle hops to its next nearest neighbor site on its right side if there is no ● or ○ in the middle. If ● and ○ meet at the same site after hopping, they annihilate instantaneously. In this process, there is no possibility that ● and ○ exist at two adjacent sites like ● ○ or ○ ● . However, ● ● or ○ ○ can exist at two adjacent sites. If all ● and ○ particles disappear, this process reaches an absorbing state. This process is the same as the original AB process [19,20] except that there exists a next nearest neighbor hopping process of a selected particle instead of a nearest neighbor hopping in the AB process. The difference of these hopping processes should not affect dynamical scaling behavior.

Let us define the density of ● and ○ particles as $\rho_\bullet = N_\bullet/L$ and $\rho_\circ = N_\circ/L$, where $N_\star$ means the total number of ★ (● or ○) particles, and L is system size. The number of ● and ○ in our system depends on the number of A and B particles. Defining $p=\rho_A$, one can easily see that the stationary active particle density $\rho_{s\bullet}$ is nonzero for $p>1/2$ and zero for $p<1/2$. Therefore, the critical point for ● is $p_c=1/2$. The dynamical behavior of ○ particles is exactly the same as that of ● particles because of the symmetric property of our model. Therefore, we will consider only the dynamical behavior of ● particles hereafter. In the AB process, where the number of A and B particles is initially the same, the density decays algebraically following the formula $\rho_\diamond(t) \sim t^{-\delta_\diamond}$ with $\delta_\diamond = d/4$ for $d<d_c(=4)$ and $\rho_\diamond(t) \sim t^{-1}$ for $d \geqslant d_c$, where $\diamond$ represents A or B, and d is the dimension of the system [19]. Therefore, we expect $\delta_\bullet = 0.25$ in our model for one dimension.

Through the numerical simulations of our model, we found that our model shows an absorbing phase transition in one dimension. We did numerical simulations of our model with $L=10^6$ sizes at the critical point in one dimension, we find the decay exponent (see Fig. 1)

$$\delta_\bullet = \beta/\nu_\parallel = 0.249(2). \tag{1}$$

We can calculate the exact value of the critical exponent β in one dimension. We can easily determine the number of active ● particles in the stationary state if we know the initially given number of A and B particles. In case of $N_A > N_B$, the stationary state density of ● particles, $\rho_{s\bullet}$, is exactly $(N_A-N_B)/L=(N_A/L-1/2)-(N_B/L-1/2)=2(p-p_c) \sim (p-p_c)$. Therefore, the value of the exponent β is 1 in one dimension. From the following relations $\delta=\beta/\nu_\parallel$ [3] and $\delta=1/4$, we have $\nu_\parallel=4$ when $d=1$.

We can also determine the exponent $\nu_\parallel$ from the behavior of the density below and above criticality by carrying out computer simulations of our model in one dimension. The density of particles follows the scaling form

$$\rho_\bullet(t,\epsilon) \sim t^{-\delta} g(t\epsilon^{\nu_\parallel}), \tag{2}$$

where $\epsilon=|p-p_c|$ denotes the distance from the critical point. By using $\delta=0.2486$ and $\epsilon=0.0001$, 0.0002, 0.0004, and 0.0128, the best collapse is obtained for $\nu_\parallel=3.8(2)$ [see Fig. 2(a)].

Since in our model there is no creation process of active particles, the number of active particles is always decreasing as time elapses. Hence, the dynamic exponent z in our model is determined only by diffusion of active particles. We expect $z=2$ in our model in one dimension [3]. Since $z=\nu_\parallel/\nu_\perp$, we also obtain $\nu_\perp=2$. Therefore, we conjecture the values of the three independent critical exponents $(\beta, \nu_\parallel, \nu_\perp)$ when $d=1$.

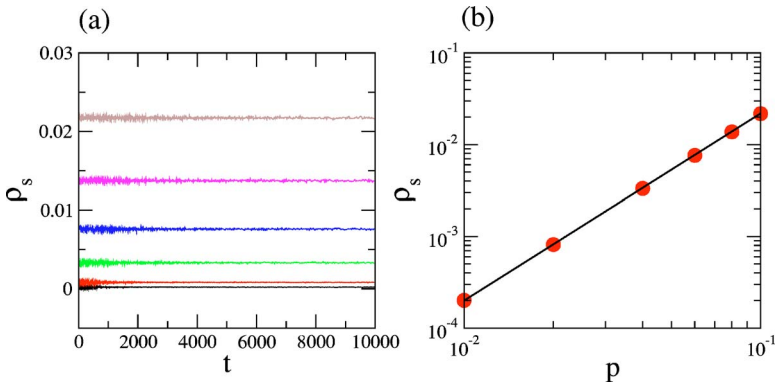

FIG. 4. (a) Stationary density $\rho_{s\bullet}$ for $p=0.01$, 0.02, 0.04, 0.06, 0.08, and 0.1 in two dimensions, averaged over 100 runs on a system with sizes $L\times L=1.6\times 10^5$. (b) The plot of $\rho_{s\bullet}$ versus p for $p=0.01$, 0.02, 0.04, 0.06, 0.08, and 0.1. The estimated slope $\beta=2.0(1)$.

We performed finite size simulations at the critical point to obtain the dynamic exponent $z=\nu_{\|}/\nu_{\perp}$ by changing the system size L from 64 to 1024. Then the density of particles $\rho_{\bullet}$ should obey the following finite-size scaling form

$$\rho_{\bullet}(t,L) \sim t^{-\delta} f(t/L^{z}), \tag{3}$$

where f is a universal scaling function. Using $\delta=0.2486$, the best collapse is obtained for $z=1.95(10)$ [see Fig. 2(b)].

Therefore, we arrive at the following result for $d=1$

$$\beta=1, \quad \nu_{\perp}=2, \quad \nu_{\|}=4. \tag{4}$$

Our model reaches an absorbing state for $p \leqslant p_c$ in one dimension, but it does not for all nonzero p when $d \geqslant 2$. In one dimension A particles surrounded by B particles cannot meet each other at all. However, although only two A particles exist in the system for $d \geqslant 2$, the two particles can meet each other by the exchange process of A and B particles as in Fig. 3. Therefore, the density of $\rho_{s\bullet}$ in a steady state is nonzero if there are more than two A particles in the system. Let us assume that there are $N_A A$ particles in a two-dimensional system with sizes $L \times L$, where $1 \ll N_A \ll L \times L$. In a given time period, the probability $P(N_A)$ for two A particles to meet is roughly proportional to $N_A(N_A-1)/2$. Since the density $\rho_{s\bullet}$ is proportional to $P(N_A)$, we obtain $\rho_{s\bullet} \sim N_A^2 \sim p^{\beta}$ with $\beta=2$. This result is in good agreement with that of the simulations [see Fig. 4(b)].

The recently introduced CLG models show a phase transition for $d>1$ unlike our model. Therefore, our model belongs to a different universality class from the CLG. This class also differs from the well known DP, PC, PCPD class as well as N-BARW2 [21,22] class, where N-BARW2 means branching and annihilating process with two offsprings in a system with N species of particles. The phase transition in N-BARW2 is associated with the blocking by different species of particles as in our model. In Table I, we summarized the critical exponents for DP, PC, PCPD, CLG model, and our model in one dimension.

TABLE I. Estimates of the critical exponents for DP, PC, PCPD, CLG model, N-BARW2, and the present model in one dimension.

Class	β	$\nu_{\perp}$	$\nu_{\|}$
DP [3]	0.2765	1.0969	1.734
PC [3]	0.92(2)	1.83(3)	3.22(6)
PCPD [10]	<0.6	1.0…1.2	1.8…2.1
CLG model [23]	1.0(1)		
N-BARW2 [24]	1	1	2
Present model	1	2	4

In conclusion, we have studied a conserved lattice gas model which exhibits a nonequilibrium absorbing phase transition with two symmetric absorbing states at criticality in one dimension. However, the model does not exhibit an absorbing phase transition in higher dimensions. In our model, the total number of particles is conserved during the temporal evolution. We have conjectured the values of the three critical exponents characterizing the phase transition of our model in one dimension. We have found that these conjectured values coincide well with those obtained from computer simulations of our model.

The authors would like to thank Professor Haye Hinrichsen for useful discussions and comments. This work was supported in part by the Korea Science and Engineering Foundation(No. R01-2004-000-10148-0) and also by the Korea Research Foundation(No. KRF-2004-005-C00060 and Brain Korea 21 projects).

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